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ON DYNAMICAL SYSTEMS IN THE PLANE***

by

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THEOREM. If (X, T) is a transformation group, T is the additive group of real numbers, X is a subspace of the plane, and x is a point of X such that $xt \neq x$ for $t \neq 0$, then $\gamma(x)$, the orbit of x, is not recurrent.*

This result is not new; it is proved in a paper written by H. Bohr and W. Fenchel in 1936 [1]. The proof we present below seems appreciably simpler than that of Bohr and Fenchel.**

NOTATION. For points y and y' of $\gamma(x)$, $y < y'$ will mean that $y = xt$ and $y' = xt'$ with $t < t'$. If y and y' are points of $\gamma(x)$ with $y < y'$, then $[y, y'] = \{z \in \gamma(x) \mid y \leq z \leq y'\}$ and $(y, y') = \{z \in \gamma(x) \mid y < z < y'\}$. If D is a closed disc in the plane, then ∂D will denote the boundary of D.

PROOF OF THEOREM. Assume that $\gamma(x)$ is recurrent. Since $xt \neq x$ for $t \neq 0$, and since disjoint closed sets in the plane possess disjoint neighborhoods, closed discs U', V', and W' can be found so that $x \in \text{Int}(V')$, $x(1) \in \text{Int}(W')$, $x(-1) \in \text{Int}(U')$, $V'[-1, 0] \cap W' = \emptyset$, $V'[0,1] \cap U' = \emptyset$. Find a closed disc $V \subset V'$ with $x \in \text{Int}(V)$, $V(1) \subset W'$, and $V(-1) \subset U'$. Set $W = W'$ and

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*In this paper the orbit of a point $x \in X$ is called recurrent if every neighborhood of x contains points xt with $t \in T$ arbitrarily large. The classical term for recurrence is stability in the sense of Poisson.

**We may mention that the theorem follows very easily if one assumes that there are local cross sections which are simple arcs. However, the proof of the existence of such cross sections is less elementary than the method we use in our proof.

***This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research, Office of Aerospace Research, under Contract No. AF 49(638)-1242, in part by the National Aeronautics and Space Administration under Contract No. NASw-845 and in part by the Office of Naval Research under Contract No. Nonr-3693(00).

(1) $$U = V(-1).$$

It follows that

(2) $$V(1) \subset W,$$

(3) $$V[-1,0] \cap W = \emptyset,$$

(4) $$V[0,1] \cap U = \emptyset.$$

Assuming $y, z \in \gamma(x)$ and $y < z$, we conclude that

(5) if $y \in U$ and $z \in W$, then $(y, z) \cap V \neq \emptyset$ [from (1) and (3)],

(6) if $y \in V$ and $z \in U$, then $(y, z) \cap W \neq \emptyset$ [from (2) and (4)], and

(7) if $y \in W$ and $z \in V$, then $(y, z) \cap U \neq \emptyset$ [from (1) and (3)].

These three implications can be summarized by saying that the orbit $\gamma(x)$ can interchange between the three sets U, V, W on y in the cyclic order $U \to V \to W \to U$.

Define points on $\gamma(x)$ as follows:

x_1 is the last point in V after x and before $\gamma(x)$ enters W.

x_3 is the last point in W after x_1 and before $\gamma(x)$ enters U.

x_4 is the first point in V after x_3.

x_6 is the first point in W after x_4.

x_2 is a point on $\partial W \cap \gamma(x)$ so that $x_1 < x_2 \leq x_3$ and there is an arc w on ∂W joining x_6 with x_2 and intersecting $[x_1, x_3]$ only in x_2.

x_5 is a point on $\partial V \cap \gamma(x)$ so that $x_4 \leq x_5 < x_6$ and there is an arc v on ∂V joining x_1 and x_5 and intersecting $[x_4, x_6]$ only in x_5. All of these points are well defined because $\gamma(x)$ is recurrent and connected; furthermore, $x_1 < x_2 \leq x_3 < x_4 \leq x_5 < x_6$ [see figure 1.].

Set $J = [x_1, x_5] \cup v$ and $J^* = [x_2, x_6] \cup w$. J and J^* are simple closed curves and $J \cap J^*$ is the arc $[x_2, x_5]$. Since $[x, x_1] \cap W = \emptyset$, and consequently $[x, x_1] \cap w = \emptyset$, x and x_1 lie in the same component of the complement of J^*; let G denote this component and H the other.

Since $x_3 \in W$ and $x_4 \in V$, we have, due to (7),

$$(x_3, x_4) \cap U \neq \emptyset. \tag{8}$$

On the other hand, it follows from (5), (6) and (7) that

$$[(J \cup J^*) - (x_3, x_4)] \cap U = \emptyset. \tag{9}$$

The region G is separated by $(x_1, x_2) \cup v$ into two sub-regions: G' bounded by $(x_1, x_5) \cup v$ and G'' bounded by $(x_1, x_2) \cup w \cup (x_5, x_6) \cup v$, [cf. figure 2.]. Due to (9), U does not meet the boundary of G''; on the other hand, due to (8), it does meet the arc (x_3, x_4), which is outside of G''. U being connected, we thus have

$$U \cap G'' = \emptyset. \tag{10}$$

Since $\gamma(x)$ is recurrent, and x is an interior point of $V \cap G$, there is an $s \in \gamma(x)$ such that $x_6 < s$ and $s \in V \cap G$. Then $(x_6, s) \cap U \neq \emptyset$, due to (7). We show:

(i) The first point of $(x_6, s) \cap U$, say p, is in H,

(ii) the last point, say q, of it is in G'. Indeed, due to (10), both p and q can be only in G' or H. Suppose $p \in G'$. Clearly $x_6 \notin \bar{G}'$. Thus $[x_6, p]$ would have to cross the boundary of G', and since $\gamma(x)$ has no double points, $[x_6, p]$ would have to cross v. But $x_6 \in$ [illegible] while $v \subset V$, so due to (7) $[x_6, p]$ would have to pass through U, which contradicts the definition of p. So (i) follows. (ii) is proved in a very similar manner, using the fact that $s \notin \bar{H}$. Now we consider the arc $[p, q]$. Since $p \in H$ and $q \in G'$, $[p, q]$ contains a last point $r \in \bar{H}$ before the orbit enters G', and a first point $r_2 \in \bar{G}'$ after r_1. Since the boundaries of H and G' can be crossed only on w and v, respectively, we have $r_1 \in w \subset W$ and $r_2 \in v \subset V$. On the other hand, $(r_1, r_2) \subset G''$, which together with (10) implies $(r_1, r_2) \cap U = \emptyset$. But now we have a contradiction with (7), so the assumption that $\gamma(x)$ is recurrent is false.

We are grateful to Joseph Auslander for conversations which resulted in this paper.

[1] H. BOHR and W. FENCHEL, Ein Satz uber stabile Bewegungen in der Ebene, Harald Bohr Collected Mathematical Words, Vol. II, C. 38, Copenhagen, The Danish Mathematical Society, 1952.

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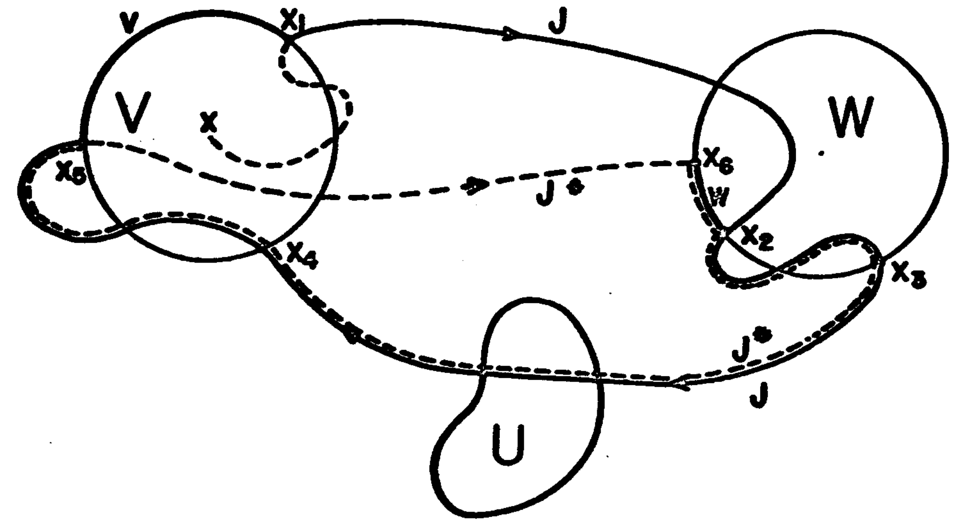

Figure 1.

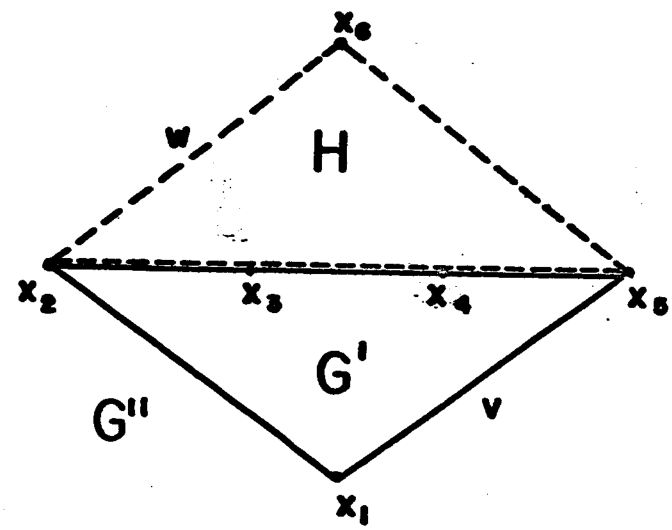

Figure 2.